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THE EQUILIBRIUM SHAPE OF A LIQUID IN A UNIFORM FORCE FIELD†

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A method of constructing the shape of the surface of a drop, hanging from a horizontal plane under the action of surface-tension forces, or lying on it or pressed between parallel planes, is proposed.

The equations of the surface of a drop are usually determined by numerical methods [1, 2] or using power series, which converge in certain neighbourhoods of the points of expansion, which subsequent analytical extension.‡ However, the equation of the surface in the first two cases indicated has an irregular singularity, which is ignored when the solution is expanded in series. This singularity is taken into account in the method proposed here.

In the case of a drop between two planes, in addition to the results obtained previously in [3] under conditions of zero gravity, an approximation of its surface by a spheroid is proposed. A certain analogy between the equations investigated and the equations of motion of a rigid body [4] is pointed out.

1. A DROP HANGING FROM A HORIZONTAL PLANE

The form of equilibrium of a liquid drop, hanging from a horizontal plane (Fig. 1) is described by the following differential equation [1]

$$\alpha(1/R_1 + 1/R_2) = \rho g y - \alpha K$$

where α is the surface tension coefficient, ρ is the density of the liquid, g is the acceleration due to gravity, R_1 and R_2 are the principal radii of curvature of the drop and K is twice the mean curvature of the drop at the base.

Choosing the quantity $\sqrt{\alpha/(\rho g)}$ as the unit of length, the equilibrium equation can be written in the following dimensionless form [1]

$$y''/(\sqrt{1+y'^2})^3 + y'/(x\sqrt{1+y'^2}) = -y + K \quad (1.1)$$

(the prime denotes a derivative with respect to x). This equation has an irregular singular point $x = 0$, which, if the non-linearity is taken into account, does not enable us to use well-known methods of integration [5].

We will seek a solution for contact angles $\theta > \pi/2$.

Introducing the new variable

$$X = y'/\sqrt{1+y'^2} \quad (1.2)$$

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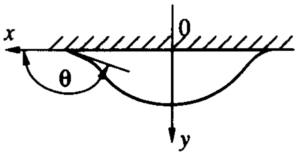

Fig. 1

we obtain from Eq. (1.1)

$$X' + X/x = -y + K \tag{1.3}$$

or, after differentiation (taking into account the fact that the signs of X and y' are the same,

$$X'' + X'/x - X/x^2 + X/\sqrt{1-X^2} = 0 \tag{1.4}$$

By virtue of the symmetry of the problem about the y axis we will seek a solution of Eq. (1.4) for $x \geq 0$ ($X \leq 0, X(0) = 0$).

In the above equation, as in (1.1), the irregularity is retained, but the non-linear part allows of an expansion in a converging power series for all values of the variable X having a physical meaning ($|X| < 1$; when $|X| = 1$ the drop breaks away from the support), which cannot be said of the quantity y' in Eq. (1.1). In addition, Eq. (1.4) enables us to draw certain qualitative conclusions. In fact, the quantities X' and X/x are the principal curvatures of the surface, which become equal to one another when $x = 0$, and hence from Eq. (1.3) we obtain

$$X'(0) = -1/R_0 = (K-h)/2, \quad h = y(0) \tag{1.5}$$

where R_0 is the radius of curvature of the meridian $R = R(x)$ when $x = 0$.

Suppose x_* and x_c are the values of x for which $X'(x_*) = 0$ and $y(x_c) = 0$ respectively. We will show that the function $R(x)$ increases monotonically from the value R_0 when x increases from 0 to x_* (or x_c if there is no point of inflection x_*).

In the neighbourhood of the point $x = 0$, Eq. (1.4) can be approximated by Bessel's equation

$$X'' + X'/x + (1 - 1/x^2) = 0 \tag{1.6}$$

Its solution, which satisfies the conditions $X(0) = 0$ and $X'(0) = -1/R_0$, is a Bessel function of the first kind [6]

$$X_B = -\frac{1}{R_0}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2^{2k}k!(k+1)!}$$

It can be seen from this solution that $X''(0) = 0$, and in the right semi-neighbourhood of the point $x = 0$ we have $X'' > 0$. If the point of the extremum of X' were to lie in the interval $(0, x_*)$, there would be one more point of the extremum of X' in this interval, since $X''(x_*) > 0$ (on passing through the value $x = x_*$ the sign of X' changes from minus to plus). Eliminating the quantity X' from Eq. (1.4) by means of relation (1.3), we obtain

$$X'' = (2/x^2) + X/\sqrt{1-X^2} + (y-K)/x \tag{1.7}$$

When x increases both terms on the right-hand side decrease monotonically, but the first one remains negative while the second one remains positive. Hence, the sum of these terms can only vanish at one point, and it is not possible for X' to vanish in the interval $(0, x_*)$.

The solution of Eq. (1.4) will be sought in the form of a power series

$$X = a_1 x + a_1 x^2 + \ldots; \quad a_1 = -1/R_0 \tag{1.8}$$

After substituting this series and its derivatives into Eq. (1.4) (taking into account the expansion of the non-linear part of the latter equation in powers of X), we obtain expressions for the coefficients in terms of the coefficient a_1

$$a_{2n} = 0, \quad a_{2n+1} = -\frac{a_{2n-1} + A_{2n-1}}{(2n+1)^2 - 1}; \quad n = 1, 2, \dots \tag{1.9}$$

$$\begin{aligned} A_{2n-1} &= \frac{(2n-3)!!}{2^{n-1}(n-1)!} a_1^{2n-1} + \frac{(2n-5)!!}{2^{n-2}(n-2)!} \sum \frac{(2n-3)!}{l_1! l_3!} a_1^{l_1} a_3^{l_3} + \dots \\ &\dots + \frac{[2(n-m)-3]!!}{2^{n-m-1}(n-m-1)!} \sum \frac{[2(n-m)-1]!}{l_1! l_3! \dots l_{2m+1}!} a_1^{l_1} a_3^{l_3} \dots a_{2m+1}^{l_{2m+1}} + \dots \\ &\dots + \frac{1}{2} \sum \frac{3!}{l_1! l_3! \dots l_{2n-3}!} a_1^{l_1} a_3^{l_3} \dots a_{2n-3}^{l_{2n-3}}, \quad n = 2, 3, \dots \end{aligned} \tag{1.10}$$

Summation is carried out over all $l_1, l_3 \dots, l_{2m+1}$, which satisfy the conditions

$$l_1 + 3l_3 + \dots + (2m+1)l_{2m+1} = 2n - 1, \quad l_1 + l_3 + \dots + l_{2m+1} = 2(n-m) - 1 \tag{1.11}$$

where $l_1, l_3 \dots, l_{2m+1}$ must represent non-negative integer solutions of the system of Diophantine equations (1.11).

Since it is difficult to determine the radius of convergence of series (1.8) directly, we will consider some estimates of the solution of Eq. (1.4).

In this equation we replace the expression $1/\sqrt{1 - X^2}$ by its first approximation $1 + X^2/2$ and, for the equation obtained

$$X'' + X'/x + (1 - X/x^2)X + X^3/2 = 0 \tag{1.12}$$

we estimate the radius of convergence of the series of the form (1.8). Then the coefficient A_{2n-1} of powers of X^{2n-1} the expansion of $X^3/2$ has the form

$$A_{2n-1} = \frac{1}{2} \sum \frac{3!}{k_1! k_3! \dots k_{2n-1}!} a_1^{k_1} a_3^{k_3} \dots a_{2n-1}^{k_{2n-1}}$$

where the summation is carried out over $k_1, k_3, \dots, k_{2n-1}$ which satisfy the system of Diophantine equations

$$k_1 + 3k_3 + \dots + (2n-1)k_{2n-1} = 2n - 1, \quad k_1 + k_3 + \dots + k_{2n-1} = 3 \tag{1.13}$$

We will later require an estimate of the number of non-negative integral solutions of this system. Corresponding estimates are only known for equations of the form

$$l_1 Z_1 + l_2 Z_2 + \dots + l_p Z_p = n \tag{1.14}$$

($l_1, l_2 \dots, l_p$ and n are non-negative integers). The number of solutions of Eq. (1.14) is equal to the coefficient E_n of the expansion [7]

$$[(1 - \xi^{l_1})(1 - \xi^{l_2}) \dots (1 - \xi^{l_p})]^{-1} = \sum_{m=0}^{\infty} E_m \xi^m \tag{1.15}$$

Applying this formula to the second equation of (1.13), we obtain

$$E_3 = n(n+1)(n+2)/6 \tag{1.16}$$

while for the first equation of (1.13), formula (1.15) is practically useless due to the complexity of calculating the coefficient E_{2n-1}. This calculation can be simplified somewhat, if we use the inequality

$$\begin{aligned} &[(1 - \xi)(1 - \xi^3)(1 - \xi^5) \dots (1 - \xi^{2n-1})]^{-1} \le (1 + \xi)(1 + \xi^2)(1 + \xi^3) \dots = \\ &= 1 + \xi + \xi^2 + 2\xi^3 + 2\xi^4 + 3\xi^5 + 3\xi^6 + 5\xi^7 + \dots \end{aligned}$$

whence

$$E_{2n+1} < n + 1 + E_n + E_{n-1} + \dots \tag{1.17}$$

Since the number of solutions of system (1.13) is much less than the number of solutions (1.16) or (1.17), for our further estimate we will change from system (1.13) to the equivalent system

$$k_1 + k_3 + k_5 + \dots + k_{2n-1} = 3, \quad k_3 + 2k_5 + 3k_7 + \dots + (n-1)k_{2n-1} = n-2; \quad n > 2 \tag{1.18}$$

We will place the solutions of this system in the following order:
$k_1 = 3, k_i = 0, i > 1$ (there will only be a solution for $n = 2$)
$k_2 = 2, k_{2n-3} = 1$, all the remaining $k_i = 0$.
We will call the following series of solutions the first series

$$k_1 = 1, \quad k_3 = 1, \quad k_{2n-5} = 1$$
$$k_1 = 1, \quad k_3 = 0, \quad k_5 = 1, \quad k_{2n-7} = 1$$
$$k_1 = 1, \quad k_3 = 0, \quad k_5 = 0, \quad k_7 = 1, \quad k_{2n-9} = 1$$

....

The number of solutions of the first series is no greater than $[n/2] - 2$.
We will consider the solutions of the second series

$$k_1 = 0, \quad k_3 = 1, \quad k_5 = 1, \quad k_{2n-9} = 1$$
$$k_1 = 0, \quad k_3 = 1, \quad k_5 = 0, \quad k_7 = 1, \quad k_{2n-11} = 1$$
$$k_1 = 0, \quad k_3 = 1, \quad k_5 = 0, \quad k_7 = 0, \quad k_9 = 1, \quad k_{2n-13} = 1$$

....

The number of solutions of this series is no greater than $[n/2] - 3$.
Since the number of these series is no greater than $[n/2] - 2$, the total number of solutions in them is no greater than $([n/2] - 1)([n/2] - 2)/2$.
We must supplement these solutions with the following:

$$k_1 = 0, \quad k_3 = 2, \quad k_{2n-7} = 1$$
$$k_1 = 0, \quad k_3 = 0, \quad k_5 = 2, \quad k_{2n-11} = 1$$

....

The number of such solutions is no greater than $[(n - 1)]/4]$.
The number of all solutions of system (1.18) is no greater than $N = 3 + [n^2/8] - [n/2]$.
To estimate the radius of convergence of the series, representing the solution of Eq. (1.12), we will assume that $|a_1| \le 1/2$. Then the following inequalities are satisfied for the first numbers $n = 1, 2, 3, 4, 5, 6$

$$|a_{2n-1}| < (1/2)^{2n-1} \tag{1.19}$$

Using this property as a basis for induction, it can be proved, using the first equality of (1.13), that it is also satisfied for $|a_{2n+1}|$. As a result we conclude that the radius of convergence of the power series considered is no less than two.

We will prove that the solution of Eq. (1.12) $\psi(x)$ is the minorant of the exact solution $\varphi(x)$ of Eq. (1.4) (when $\psi(0) = \varphi(0) = 0, \psi'(0) = \varphi'(0) = a_1$).

A comparison of the solutions $\varphi(x) = a_1x + a_3x^3 + a_5x^5 + \dots$ and $\psi(x) = a_1x + \bar{a}_3x^3 + \bar{a}_5x^5 + \dots$ in a small neighbourhood of $x = 0$ shows that $\varphi(x) \ge \psi(x)$ (and $\varphi'(x) \ge \psi'(x)$), since $\bar{a}_3 = a_3, \bar{a}_5 = a_5, \bar{a}_7 < a_7$.

We will now prove that the inequality $\varphi(x) \ge \psi(x)$ remains valid in the interval of convergence of the series representing $\psi(x)$.

We will write Eqs (1.4) and (1.12) respectively in the form

$$(xX')' = (1/x - x)X - x(X^3/2 + 3X^5/8 + \dots), \quad (xX')' = (1/x - x)X - xX^3/2$$

We substitute the corresponding solutions into these equations and, subtracting one from the other, we obtain

$$[x(\varphi' - \psi')]' = (1/x - x)(\varphi - \psi) - x/2(\varphi^3 - \psi^3) - x(3\varphi^5/8 + \dots)$$

We will assume that we have obtained the point $x_1 \in (0; 1)$, at which the curve $\varphi(x)$ and $\psi(x)$ intersect, i.e. $\varphi(x_1) = \psi(x_1)$ and $\varphi'(x_1) < \psi'(x_1)$ and for $x \in (x_1; x_1 + \varepsilon), \varepsilon > 0$ we have $\varphi(x) < \psi(x)$ and $\varphi'(x) < \psi'(x)$. Then, after integrating the last equation over a fairly small section $[x_1, x_1 + \varepsilon]$ from the left and from the right we obtain a number of different signs. This is absurd. Hence, intersection of the curves is impossible.

Together with the minorant $\psi(x)$ for the solution $\varphi(x)$ we can indicate its majorant $\Psi(x)$ if we replace $1/\sqrt{1-X^2}$ in Eq. (1.4) by a portion of a parabola $1 + \lambda X^2$, by choosing the quantity λ such that for $0 \le X^2 \le X_*^2 < 1$ the inequality $1/\sqrt{1-X^2} \le 1 + \lambda X^2$ is satisfied. For example, $\lambda = 0.62$ when $X_* = -1/2$. With this replacement, the majorant $\Psi(x)$ is defined in the form of a series, similar to the definition of $\psi(x)$.

Hence, although the radius of convergence of series (1.8) has not been established, this series can be used, at least, for values of x for which the numerical values of the series remain between the values $\psi(x)$ and $\Psi(x)$, and we thereby have an estimate of the error.

Using formulae (1.9) and (1.10) we can write the first coefficients of the expansion (1.8) in the explicit form

$$a_3 = -\frac{1}{8}a_1, \quad a_5 = -\frac{1}{24}\left(\frac{1}{2}a_1^3 + a_3\right), \quad a_7 = -\frac{1}{48}\left(\frac{3}{8}a_1^5 + \frac{3}{2}a_1^2a_3 + a_5\right)$$

$$a_9 = -\frac{1}{80}\left(\frac{5}{16}a_1^7 + \frac{15}{8}a_1^4a_3 + \frac{3}{2}a_1^2a_5 + \frac{3}{2}a_1a_3^2 + a_7\right)$$

From relation (1.3) we have the solution

$$y = K - X' - X/x = K - 2a_1 - 4a_3x^2 - 6a_5x^4 - \ldots \tag{1.20}$$

which contains two parameters: K and R_0. In order to determine these parameters, we will use expression (1.2) and the boundary condition $y'(x_c) = \operatorname{tg}\theta$; from expression (1.2) we then obtain $X(x_c, R_0) = -\sin\theta$. From Eq. (1.3) we express $K = X'(x_c, R_0) - (\sin\theta)/x_c$ and, substituting into relation (1.20) (for $x = x_c$ and $y = 0$), we express $x_c = x_c(R_0)$ and $K = K(R_0)$ and $y = y(x, R_0)$.

The volume of the drop $V = 2\pi \int_0^{x_c} yxdx$ (assumed given) determines the values of R_0.

Below, for values of $R_0 = 1$ and $h = 1$, we present the results of a calculation of y from formula (1.2) and the quantity Δ, equal to the difference in the values of y obtained and the results of a numerical solution of Eq. (1.1)

x	0	0.2	0.4	0.6	0.8	1
y	1	0.9978	0.9433	0.8495	0.6988	0.4690
$\Delta \cdot 10^4$	0	0	0	0	1	15

2. A DROP ON A PLANE

We will consider the shape of a drop lying on a plane (Fig. 2) and having a contact angle $\theta > \pi/2$. The equation of the meridian in this case will differ from (1.1) solely in the sign of the right-hand side (the direction of the y axis is now opposite to the direction of the force of gravity) and instead of Eq. (1.4) we obtain an equation in which the sign of the last term is changed from plus to minus. In the neighbourhood of the point $x = 0$ in the linear approximation it has the form of an equation related to Bessel's equation [6].

We will again seek a solution of the problem in the form of series (1.8). We now obtain

$$a_{2n} = 0, \quad a_{2n+1} = \frac{a_{2n-1} + A_{2n-1}}{(2n+1)^2 - 1}$$

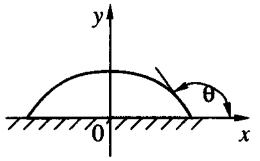

Fig. 2

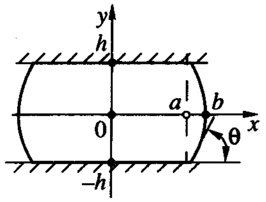

Fig. 3

(the quantities A_{2n-1}, as before, are defined by formulae (1.10)). Further, instead of expression (1.20) we have

$$y = K + X/x + X' = K + 2a_1 + 4a_3x^2 + \ldots$$

3. A DROP BETWEEN PARALLEL PLANES UNDER ZERO GRAVITY CONDITIONS

We will consider the shape of a drop, confined between parallel walls (Fig. 3) in zero gravity conditions with contact angles $\theta < \pi/2$ (unwetted). Suppose the y axis coincides with the axis of symmetry of the drop, while the x axis lies in the plane of its equator at a distance h from the wall. The equation for the variable X takes the form

$$X' + X/x = -K \quad (K > 0) \tag{3.1}$$

and its general solution is

$$X = -Kx/2 + c_1/x \tag{3.2}$$

Although the solution for y is expressed in quadratures, the exact solution of the boundary-value problem is quite complex to obtain even for small deformations of the drop [3].

We will consider an approximation of the meridian of the drop by an arc of an ellipse, using the monitoring of a change in the curvature of the meridian. Suppose a and b are the values of x for which the meridian comes in contact with the wall and intersects the equator of the drop, respectively. Using the boundary condition

$$X(b) = -1 = -Kb/2 + c_1/b$$

we obtain

$$c_1 = (Ka/2 - \sin\theta)a \tag{3.3}$$

It can be seen that

$$c_1 > 0 \text{ and } c = -2(b + a\sin\theta)/(b^2 - a^2) \tag{3.4}$$

It follows from relations (3.1) and (3.2) that the curvature of the meridian decreases monotonically in modulus as x increases.

When $x = a$ and $y = h$ the radius of the principal curvature along a parallel is equal to $a/\sin\theta$. Hence, the radius of curvature of the meridian at the point of contact is

$$R_a = 1/|X'(a)| = a/(aK - \sin\theta)$$

At the point of intersection with the equator, the radius of curvature of the meridian $R_0 = -b(1 - Kb)$ (since the equator is a line of curvature).

We will approximate the meridian by the arc of an ellipse so that the latter will have the specified contact angle θ and a curvature which decreases in modulus as x increases, where its radius of curvature ρ at the point $x = a$, $y = h$ is identical with R_a, while the radius of curvature ρ_0 at the point $(b, 0)$ is identical with R_0.

Suppose $\xi^2/A^2 + y^2/B^2 = 1$ is the required equation of the ellipse, where $\xi = x - a - l$, where l is the distance from the centre of the ellipse to the point $x = a$ and $\rho(y, \xi)$ is the radius of curvature of the ellipse. Here

$$\rho(y, \xi) = \frac{1}{AB}\left(\frac{A^4y^2 + B^4\xi^2}{A^2y^2 + B^2\xi^2}\right)^{3/2}$$

Choosing the parameters A, B and l so that the above requirements are satisfied, we obtain

$$l = \frac{A^2\operatorname{tg}\theta}{\sqrt{A^2\operatorname{tg}^2\theta + B^2}}, \quad \frac{B^2}{A} = \frac{-b}{1 - Kb}, \quad \rho(h, l) = \frac{a}{aK - \sin\theta}$$

The boundary conditions

$$b = a - A - l, \quad h = B\sqrt{1 - l^2/A^2}$$

and the condition for the volume of the drop to be constant

$$V = 2\pi a^2 h + 2\pi\int_a^b yx dx = 2\pi a^2 h +$$

$$+\pi(a-l)AB\left(\frac{\pi}{2} - \arcsin\frac{l}{A} - \frac{l}{A}\sqrt{1-\frac{l^2}{A^2}}\right) + \frac{2}{3}\pi A^2 B\left(\sqrt{1-\frac{l^2}{A^2}}\right)^3$$

are connected with these relations.

In particular, for complete wetting ($\theta = 0$) we obtain

$$l = 0, \quad B = h, \quad V = 2\pi a^2 h + (\pi^2/2)Aha + 2\pi A^2 h/3$$

Hence, determining the quantity a as the positive root $a = a(A)$, from the previous formulae we find $K = h/A^2$ and $A = A(h)$.

When $\theta > 0$ the values of the parameters obtained can be used as the first approximations for calculating the corrections.

Remark. In the problem of reducing the order of the differential equations of motion of a rigid body ([4] etc.) the following equation is obtained

$$\frac{y''}{(\sqrt{1+y'^2})^3} = \frac{(V_y - V_x y')^2}{2V\sqrt{1+y'^2}} \pm \frac{\Omega}{\sqrt{2V}}$$

where V and Ω are certain specified functions of the variables x and y; $V_x = \partial V/\partial x$, $V_y = \partial V/\partial_y$.

This equation bears some similarity to the equation considered above. By replacing the variable $X = 1/\sqrt{1+y^2}$ it can be reduced to the system

$$y' = \frac{\sqrt{1+X^2}}{X}, \quad X' = \frac{1}{2}\left(V_y - \frac{V_x\sqrt{1-X^2}}{X}\right)^2\frac{\sqrt{1-X^2}}{V} \pm \frac{\Omega}{X}\sqrt{\frac{1-X^2}{2V}}$$

the right-hand sides of which can be represented in the form of the products of series in powers of X^2 in X^{-1} or X^{-2} over the whole range of values of $X \in [0, 1)$.

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